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On Barnes' Multiple Zeta and Gamma Functions

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We show how various known results concerning the Barnes multiple zeta and gamma functions can be obtained as specializations of simple features shared by a quite extensive class of functions. The pertinent functions involve Laplace transforms, and their asymptotics is obtained by exploiting this. We also demonstrate how Barnes' multiple zeta and gamma functions fit into a recently developed theory of minimal solutions to first order analytic difference equations. Both of these new approaches to the Barnes functions give rise to novel integral representations.

Contents.

1. INTRODUCTION

In an impressive series of papers [1–4] culminating in Ref. [5], Barnes developed a comprehensive theory for a new class of special functions, the so-called multiple zeta and gamma functions. Barnes' multiple zeta function $\zeta_N(s, w \mid a_1, ..., a_N)$ depends on parameters $a_1, ..., a_N$ that will be taken positive throughout this paper. It may be defined by the series

$$\zeta_N(s, w \mid a_1, ..., a_N) = \sum_{m_1, ..., m_N = 0}^{\infty} (w + m_1 a_1 + \cdots + m_N a_N)^{-s}, \quad \text{Re } w > 0, \ \text{Re } s > N, \qquad (1.1)$$

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from which the recurrence relation

$$\begin{aligned}\zeta_{M+1}(s, w + a_{M+1} \mid a_1, ..., a_{M+1})\\ -\zeta_{M+1}(s, w \mid a_1, ..., a_{M+1}) = -\zeta_M(s, w \mid a_1, ..., a_M)\end{aligned} \tag{1.2}$$

is immediate (with $\zeta_0(s, w) = w^{-s}$).

Barnes showed that ζ_N has a meromorphic continuation in s, with simple poles only at $s = 1, ..., N$, and defined his multiple gamma function $\Gamma_N^B(w)$ in terms of the s-derivative at $s = 0$, which we will write

$$\Psi_N(w \mid a_1, ..., a_N) = \partial_s \zeta_N(s, w \mid a_1, ..., a_N)|_{s=0}. \tag{1.3}$$

Clearly, analytic continuation of (1.2) yields the recurrence

$$\begin{aligned}\Psi_{M+1}(w + a_{M+1} \mid a_1, ..., a_{M+1})\\ -\Psi_{M+1}(w \mid a_1, ..., a_{M+1}) = -\Psi_M(w \mid a_1, ..., a_M),\end{aligned} \tag{1.4}$$

with $\Psi_0(w) = -\ln w$.

Up to inessential factors, the functions ζ_1 and Ψ_1 are equal to the Hurwitz zeta function and the logarithm of Euler's gamma function (cf., e.g., Ref. [6]). For $a_1 = a_2 = 1$, the function

$$S_2(w \mid a_1, a_2) = \exp(\Psi_2(a_1 + a_2 - w \mid a_1, a_2) - \Psi_2(w \mid a_1, a_2)) \tag{1.5}$$

was already studied by Hölder in 1886 [7]. It was called the double sine function by Kurokawa. More generally, Kurokawa considered multiple sine functions defined in terms of $\Psi_N(w)$, relating these functions to Selberg zeta functions and determinants of Laplacians occurring in symmetric space theory [8–10]. (See Refs. [11–13] for earlier work in this direction.)

Barnes' multiple zeta and gamma functions were also encountered by Shintani within the context of analytic number theory [14, 15]. In recent years, they showed up in the form factor program for integrable field theories [16, 17] and in studies of XXZ model correlation functions [18]. See also recent papers by Nishizawa and Ueno [19–21], where q-analogs of the multiple gamma functions are studied.

In our lectures on Calogero–Moser type systems [22] we introduced a function that is substantially equal to the double sine function (1.5). We dubbed it the hyperbolic gamma function, for reasons made clear in our paper Ref. [23]. (Only recently we became aware of the connections to the previous work by Barnes, Shintani and Kurokawa, as detailed in Appendix A of Ref. [24].) From the viewpoint expounded in Ref. [23], the hyperbolic gamma function (alias double sine function) is a solution to a first order analytic difference equation with properties that render it unique. Informally, these properties amount to its having the maximal analyticity and mildest increase at infinity that is compatible with the difference equation.

As it turns out, the theory of first order analytic difference equations developed in Ref. [23] naturally applies to Barnes' multiple zeta and gamma functions. (In Appendix A of Ref. [23] we already detailed how Euler's gamma function fits in.) Indeed, a principal goal of this paper is to make clear in what sense ζ_{M+1} and Ψ_{M+1} may be viewed as the simplest solution to the equations (1.2) and (1.4), interpreted as analytic difference equations for unknown functions, with the right-hand sides ζ_M and Ψ_M being regarded as explicitly given functions. (In fact, Barnes used this expression, without going beyond an intuitive notion of simplicity.)

Within our framework, the idea of the simplest solution is replaced by the precisely defined concept of a *minimal solution*. We have summarized the pertinent results from Ref. [23] in Appendix A, where we also present two new results (Theorems A.2 and A.3) that are relevant in the present setting. The application to the special difference equations (1.2) and (1.4) is studied in Section 4. (Accordingly, the reader is advised to glance at Appendix A before reading Section 4.) It leads to useful new representations for ζ_N and Ψ_N, of which we mention specifically the remarkable formula

$$\zeta_N\left(s, \sum_{j=1}^{N} a_j/2 + w\right) = \int_{\mathbb{R}^N} \left(\prod_{n=1}^{N} \frac{\pi \operatorname{ch}^{-2}(\pi x_n/a_n)}{2a_n^2(s-n)} \right) \left(w - i \sum_{n=1}^{N} x_n \right)^{N-s} dx_1 \cdots dx_N, \tag{1.6}$$

cf. (4.13). Indeed, it is immediate from this representation that ζ_N admits a meromorphic continuation in s, with simple poles for $s = 1, \ldots, N$, and the s-derivative at $s = 0$ can be readily calculated from this formula as well.

As his main tool to handle s-continuation and derive large-w asymptotics, Barnes [5] employed a representation in terms of contour integrals, generalizing the Hankel integral representation for the gamma function (see, e.g., Ref. [6]). A second goal of this paper is to show how these aspects can be quite easily dealt with for a very general class of functions, using Laplace transforms as the main tool. (Barnes' arguments yielding the large-w asymptotics (cf. Section 57 in Ref. [5]) are quite involved; Shintani's Proposition 4 in Ref. [14] dealing with the double gamma function does not simplify matters either.)

Section 2 is devoted to this general setup. It is quite independent of the difference equation theory in Appendix A, and leads to representations that are different from the formulas arising in the difference equation framework. On the other hand, we have occasion to invoke a general result on the asymptotics of certain Laplace transforms, which we arrived at and applied in the difference equation context of Ref. [23]. Save for this result (Theorem B.1 in Ref. [23]), Section 2 is self-contained and quite elementary, involving solely some well-known properties of Euler's gamma function.

In Section 3 we focus attention on the special functions that yield the Barnes zeta and gamma functions. Thus we quickly arrive at a substantial part of the results obtained by Barnes. (In particular, almost all of the formulas in the Jimbo–Miwa summary on Barnes' functions arise in this way, cf. Appendix A in Ref. [18].) Moreover, we are led to new representations that are quite different from the Hankel type representations occurring in Barnes' papers and later work.

The difference equation viewpoint explained in Section 4 (and the alternative representations to which it leads) might be exploited to quickly reobtain some other results due to Barnes. In particular, his transformation theory (cf. Sections 45–48 in Ref. [5]) may be arrived at by taking the general addition formula (A.9) as a starting point. But the main purpose of this paper is to present a concise and largely self-contained account of some highlights among Barnes' results, supplying in the process novel representations and the minimal solution interpretation that may be useful for further studies and applications of the Barnes functions.

2. GENERALIZED BARNES FUNCTIONS

Let $f(t)$ be a continuous function on $[0, \infty)$ with at worst polynomial growth as $t \to \infty$. Choosing $\ln t$ real on $(0, \infty)$, we begin by studying the integral (Mellin–Laplace transform)

$$\int_0^\infty \frac{dt}{t} \exp(z \ln t - wt) f(t) \equiv F(z, w). \tag{2.1}$$

It is easily verified that $F(z, w)$ is a well-defined analytic function for

$$(z, w) \in \{\operatorname{Re} z > 0\} \times \{\operatorname{Re} w > 0\}, \tag{2.2}$$

which satisfies

$$\partial_w F(z, w) = -F(z+1, w). \tag{2.3}$$

From now on we assume that there exist $\alpha_k \in \mathbb{C}$, $k \in \mathbb{N}$, such that for all $l \in \mathbb{N}$ one has

$$f(t) - \sum_{k=0}^{l} \frac{\alpha_k t^k}{k!} = O(t^{l+1}), \qquad t \downarrow 0. \tag{2.4}$$

This enables us to associate Bernoulli-like polynomials with the function f, as follows:

$$\mathscr{B}_n(x) \equiv \sum_{k=0}^{n} \binom{n}{k} \alpha_k x^{n-k}, \qquad n \in \mathbb{N}. \tag{2.5}$$

Indeed, this definition entails the Bernoulli type features

$$\alpha_l = \mathscr{B}_l(0), \quad \mathscr{B}'_{l+1}(x) = (l+1)\,\mathscr{B}_l(x), \qquad \forall l \in \mathbb{N}. \tag{2.6}$$

We are now prepared for our first proposition.

PROPOSITION 2.1. *Fixing w with* $\operatorname{Re} w > 0$, *the function $g_w(z) = F(z, w)$ extends to a function that is holomorphic for $z \notin -\mathbb{N}$. For $z = -n$, $n \in \mathbb{N}$, the function $g_w(z)$ has a simple pole with residue $\mathscr{B}_n(-w)/n!$.*

Proof. We have

$$\int_0^\infty \frac{dt}{t} t^{z+l} e^{-wt} = w^{-z-l} \int_0^\infty \frac{du}{u} u^{z+l} e^{-u} = w^{-z-l}\,\Gamma(z+l). \tag{2.7}$$

Fixing $M \in \mathbb{N}$, we therefore obtain

$$F(z, w) = \sum_{k=0}^{M} \frac{\alpha_k}{k!} w^{-z-k}\,\Gamma(z+k) + \int_0^\infty \frac{dt}{t} t^z e^{-wt} \left(f(t) - \sum_{k=0}^{M} \frac{\alpha_k t^k}{k!} \right). \tag{2.8}$$

Now the term in brackets is $O(t^{M+1})$ for $t \downarrow 0$, so the integral yields a function that is analytic for $\operatorname{Re} z > -M-1$. The remaining terms have simple poles for $z + k \in -\mathbb{N}$. Therefore it remains to verify the residue assertion. To this end we need only recall that the residue of the function $\Gamma(s)$ at its pole $s = -m$ is given by $(-)^m/m!$. ∎

We proceed by associating a generalized multiple zeta function with the function f:

$$Z_N(s, w) \equiv F(s-N, w)/\Gamma(s). \tag{2.9}$$

Fixing $M \geqslant N$, we obtain from (2.8) the representation

$$\begin{aligned} Z_N(s,w) = {} & \sum_{k=0}^{N} \frac{\alpha_k}{k!} w^{N-s-k} \prod_{l=1}^{N-k} \frac{1}{s-l} \\ & + \sum_{k=N+1}^{M} \frac{\alpha_k}{k!} w^{N-s-k} \prod_{l=0}^{k-N-1} (s+l) \\ & + \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{dt}{t} t^{s-N} e^{-wt} \left(f(t) - \sum_{k=0}^{M} \frac{\alpha_k t^k}{k!} \right). \qquad (2.1 \end{aligned}$$

PROPOSITION 2.2. *For fixed* w *with* $\mathrm{Re}\, w > 0$ *the function* $Z_N(s,w)$ *holomorphic for* $s \notin \{1, ..., N\} \equiv \mathscr{P}_N$, *and for fixed* s *with* $s \notin \mathscr{P}_N$ *it holomorphic in* $\mathrm{Re}\, w > 0$. *It satisfies*

$$\partial_w^M Z_N(s,w) = (-)^M s(s+1) \cdots (s+M-1) Z_N(s+M, w), \qquad M \in \mathbb{N}^* \qquad (2.1$$

and

$$Z_N(-m, w) = (-)^m \frac{m!}{(N+m)!} \mathscr{B}_{N+m}(-w), \qquad m \in \mathbb{N}. \qquad (2.1$$

At $s = j \in \mathscr{P}_N$ *it has a simple pole with residue*

$$r_j = \frac{1}{(j-1)!\,(N-j)!} \mathscr{B}_{N-j}(-w), \qquad j \in \{1, ..., N\}. \qquad (2.1$$

Proof. Clearly, (2.11) follows from (2.3) and (2.9). The remaining asse tions follow from (2.9) and Prop. 2.1. (Alternatively, they can be deduc directly from the representation (2.10).) ∎

Next, we introduce a function

$$L_N(w) \equiv \partial_s Z_N(s,w)|_{s=0}, \qquad (2.1$$

which may be viewed as the logarithm of a generalized multiple gamn function associated with f. From (2.10) we obtain the representation

$$\begin{aligned} L_N(w) = {} & \sum_{k=0}^{N} \frac{\alpha_k (-w)^{N-k}}{k!\,(N-k)!} \left(\sum_{l=1}^{N-k} \frac{1}{l} - \ln w \right) \\ & + \sum_{k=N+1}^{M} \frac{\alpha_k}{k!} w^{N-k} (k-N-1)! + R_M(w). \qquad (2.1 \end{aligned}$$

where

$$R_M(w) \equiv \int_0^{\infty} \frac{dt}{t}\, t^{-N} e^{-wt}\left(f(t) - \sum_{k=0}^{M} \frac{\alpha_k t^k}{k!} \right), \qquad M \geqslant N. \tag{2.16}$$

Moreover, from (2.11) we deduce

$$\partial_w^M L_N(w) = (-)^M (M-1)!\, Z_N(M, w), \qquad M \geqslant N+1. \tag{2.17}$$

From now on we assume $f(t)$ is analytic for $\operatorname{Re} t > 0$ and at $t = 0$. Thus we have

$$\alpha_k = f^{(k)}(0) \tag{2.18}$$

and there exists $\delta > 0$ such that

$$f(t)\, e^{xt} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{B}_n(x), \qquad |t| < \delta. \tag{2.19}$$

Moreover, we assume that for all $k \in \mathbb{N}$, $\varepsilon > 0$ and $\chi \in [0, \pi/2)$ one has bounds

$$|f^{(k)}(re^{i\phi})| \leqslant c_{\varepsilon,k}(\chi)\, e^{\varepsilon r}, \qquad \forall (r, \phi) \in [0, \infty) \times [-\chi, \chi], \tag{2.20}$$

where $c_{\varepsilon,k}(\chi)$ is a positive non-decreasing function on $[0, \pi/2)$.

PROPOSITION 2.3. *Fixing $M \geqslant N$, the function $R_M(w)$ has an analytic continuation to*

$$\mathbb{C}^- \equiv \mathbb{C} \setminus (-\infty, 0]. \tag{2.21}$$

Fixing $\varepsilon > 0$, $\chi \in [0, \pi/2)$ and $K > \varepsilon$, one has

$$|w^{M-N} R_M(w)| \leqslant C_\varepsilon(\chi)(K - \varepsilon)^{-1}, \qquad \forall w \in S_{K,\chi}, \tag{2.22}$$

where

$$S_{K,\chi} \equiv \bigcup_{|\phi| \leqslant \chi} \{\operatorname{Re}(e^{i\phi} w) \geqslant K\}, \tag{2.23}$$

and where $C_\varepsilon(\chi)$ is a positive non-decreasing function on $[0, \pi/2)$ (Fig. 1).

Proof. Consider the function

$$f_M(t) \equiv t^{-N-1}\left(f(t) - \sum_{k=0}^{M} \frac{\alpha_k t^k}{k!} \right). \tag{2.24}$$

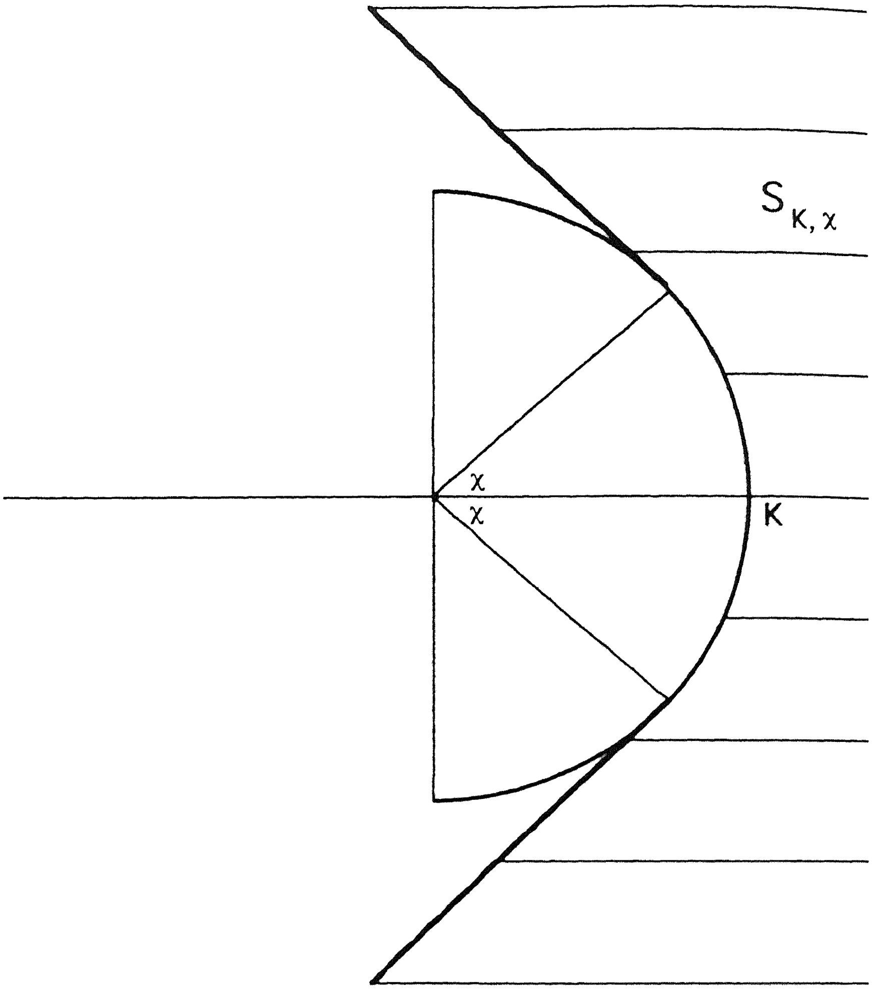

FIG. 1. The region $S_{\kappa,\chi}$.

At $t=0$ it is analytic and has a zero of order $M-N$. Thus, we can integrate by parts $M-N$ times in the representation

$$R_M(w) = \int_0^\infty dt\, e^{-wt} f_M(t), \qquad \operatorname{Re} w > 0, \tag{2.25}$$

to obtain

$$R_M(w) = w^{N-M} \int_0^\infty dt\, e^{-wt} h(t), \tag{2.26}$$

where we have set

$$h(t) \equiv f_M^{(M-N)}(t). \tag{2.27}$$

Now the function $h(t)$ is analytic for $\operatorname{Re} t > 0$ and at $t = 0$. Moreover, in view of the bounds (2.20) it satisfies for all $\varepsilon > 0$ and $\chi \in [0, \pi/2)$ a bound of the form

$$|h(re^{i\phi})| \leqslant C_\varepsilon(\chi)\, e^{\varepsilon r}, \qquad \forall (r, \phi) \in [0, \infty) \times [-\chi, \chi]. \tag{2.28}$$

Thus the assertion follows from Theorem B.1 in Ref. [23]. ▮

As an obvious corollary, we deduce that $L_N(w)$ has a holomorphic extension to $\mathbb{C}^-$. The representation (2.15), combined with the bound (2.22), now yields an asymptotic expansion that is uniform as $|w| \to \infty$ in sectorial regions $|\arg w| \leqslant \pi - \delta$, $\delta > 0$. To illustrate why this is the case, we have added Fig. 1, which depicts the geometric state of affairs.

Next, we point out that when $f(t)$ satisfies the above assumptions, so does

$$f_d(t) \equiv e^{-dt} f(t), \qquad \operatorname{Re} d > 0. \tag{2.29}$$

Specifically, $f_d(t)$ is analytic for $\operatorname{Re} t > 0$ and at $t = 0$, and $f_d(t)$ obeys the bounds (2.20). Moreover, we may take $\varepsilon = 0$ in the latter and hence in (2.22), too. Of course, the functions $Z_{N,d}(s, w)$ and $L_{N,d}(w)$ associated to f_d fulfil

$$Z_{N,d}(s, w) = Z_N(s, w + d), \qquad L_{N,d}(w) = L_N(w + d), \tag{2.30}$$

but it should be stressed that these relations are not manifest from the above representations for $Z_{N,d}$ and $L_{N,d}$.

A quite simple, yet illuminating example illustrating the latter remark and the above constructions is obtained by taking $f(t) = 1$. Obviously, f satisfies all assumptions, and (2.15) yields

$$L_N(w) = \frac{(-w)^N}{N!} \left(\sum_{l=1}^{N} \frac{1}{l} - \ln w \right) \qquad (f(t) = 1). \tag{2.31}$$

The Bernoulli polynomials associated to $f_d(t) = e^{-dt}$ are given by $\mathscr{B}_n(x) = (x - d)^n$, cf. (2.19). Taking $M = N$ in the representation (2.15) of $L_{N,d}(w)$, we obtain the identity

$$\frac{(-w-d)^N}{N!}\left(\sum_{l=1}^{N}\frac{1}{l}-\ln(w+d)\right)=-\frac{1}{N!}(-w-d)^N\ln w$$
$$+\sum_{k=0}^{N-1}\frac{(-d)^k(-w)^{N-k}}{k!(N-k)!}\sum_{l=1}^{N-k}\frac{1}{l}$$
$$+\int_0^\infty\frac{dt}{t}t^{-N}e^{-wt}\left(e^{-dt}-\sum_{k=0}^{N}\frac{(-dt)^k}{k!}\right). \tag{2.32}$$

Now when we write

$$-\frac{1}{N!}(-d)^N\ln w=-\frac{1}{N!}(-d)^N\int_0^\infty\frac{dt}{t}(e^{-t}-e^{-wt}) \tag{2.33}$$

on the rhs of (2.32), we obtain the $w\downarrow 0$ limit

$$\int_0^\infty\frac{dt}{t^{N+1}}\left(e^{-dt}-\sum_{j=0}^{N-1}\frac{(-dt)^j}{j!}-\frac{(-dt)^N}{N!}e^{-t}\right)=\frac{(-d)^N}{N!}\left(\sum_{l=1}^{N}\frac{1}{l}-\ln d\right), \tag{2.34}$$

where $\operatorname{Re} d>0$.

To conclude this section, we point out that the integral we have just derived can be exploited to rewrite $L_N(w)$ (2.15) as a single integral. Indeed, taking $M=N$ in (2.15) and using (2.34) with $d\to w$, $N\to N-k$, we obtain the integral representation

$$L_N(w)=\int_0^\infty\frac{dt}{t^{N+1}}\left(e^{-wt}f(t)-\sum_{n=0}^{N-1}\frac{t^n}{n!}\mathscr{B}_n(-w)-\frac{t^Ne^{-t}}{N!}\mathscr{B}_N(-w)\right). \tag{2.35}$$

(Recall (2.19) in order to appreciate the integrand.)

3. BARNES' MULTIPLE ZETA AND GAMMA FUNCTIONS

In order to specialize the above to the Barnes functions, we need to choose a function f that depends on the integer N we have fixed in the previous section. Specifically, we need the choice

$$f(t)=t^N\prod_{j=1}^{N}(1-e^{-a_jt})^{-1}, \qquad a_1,\ldots,a_N\in(0,\infty). \tag{3.1}$$

Clearly, this function satisfies all of our assumptions in Section 2: It is polynomially bounded for $t\uparrow\infty$, analytic for $\operatorname{Re} t>0$ and at $t=0$, and it obeys the bounds (2.20). We denote Z_N and L_N with this choice of f by $\zeta_N(s, w)$ and $\Psi_N(w)$.

Accordingly, the (Barnes) multiple zeta function reads (cf. (2.9))

$$\zeta_N(s, w) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s e^{-wt} \prod_{j=1}^N (1 - e^{-a_j t})^{-1}, \quad \mathrm{Re}\, s > N, \ \mathrm{Re}\, w > 0. \tag{3.2}$$

It can be rewritten as a power series by using

$$\prod_{j=1}^N (1 - e^{-a_j t})^{-1} = \sum_{m_1, \ldots, m_N = 0}^\infty \exp(-t(m_1 a_1 + \cdots + m_N a_N)) \tag{3.3}$$

and the integral (2.7) (with $l = 0$). This yields the formula

$$\zeta_N(s, w) = \sum_{m_1, \ldots, m_N = 0}^\infty (w + m_1 a_1 + \cdots + m_N a_N)^{-s}, \qquad \mathrm{Re}\, s > N, \ \mathrm{Re}\, w > 0, \tag{3.4}$$

mentioned in the Introduction, which is used as a starting point by Barnes [5].

In order to relate the Bernoulli-type polynomials $\mathscr{B}_n(x)$ associated with f (3.1) (cf. (2.4) and (2.5)) to the so-called multiple Bernoulli polynomials $B_{N,n}(x)$ defined by

$$\frac{t^N e^{xt}}{\prod_{j=1}^N (e^{a_j t} - 1)} = \sum_{n=0}^\infty \frac{t^n}{n!} B_{N,n}(x), \tag{3.5}$$

we exploit the identity (cf. (2.19))

$$\sum_{n=0}^\infty \frac{(-t)^n}{n!} \mathscr{B}_n(-x) = f(-t)\, e^{xt} = \frac{t^N e^{xt}}{\prod_{j=1}^N (e^{a_j t} - 1)}. \tag{3.6}$$

Indeed, a comparison yields

$$\mathscr{B}_n(x) = (-)^n B_{N,n}(-x), \qquad \alpha_n = (-)^n B_{N,n}(0). \tag{3.7}$$

Correspondingly, the general formula (2.10) specializes to

$$\begin{aligned} \zeta_N(s, w) = &\sum_{k=0}^N \frac{(-)^k}{k!} B_{N,k}(0)\, w^{N-s-k} \prod_{l=1}^{N-k} \frac{1}{s-l} \\ &+ \sum_{k=N+1}^M \frac{(-)^k}{k!} B_{N,k}(0)\, w^{N-s-k} \prod_{l=0}^{k-N-1} (s+l) \\ &+ \frac{1}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s e^{-wt} \left(\prod_{j=1}^N (1 - e^{-a_j t})^{-1} - \sum_{k=0}^M \frac{(-)^k}{k!} B_{N,k}(0)\, t^{k-N} \right), \end{aligned} \tag{3.8}$$

where $M \geqslant N$ and $\mathrm{Re}\, s > N - M - 1$. Moreover, from Prop. 2.2 we deduce that $\zeta_N(s, w)$ has a meromorphic extension with simple poles at $s \in \mathscr{P}_N$, whose residues read

$$r_j = \frac{(-)^{N-j}}{(j-1)!\,(N-j)!} B_{N,N-j}(w), \qquad j \in \{1, \ldots, N\}. \tag{3.9}$$

The values at $s = -m$, $m \in \mathbb{N}$, are given by

$$\zeta_N(-m, w) = \frac{(-)^N m!}{(N+m)!} B_{N,N+m}(w), \qquad m \in \mathbb{N}, \tag{3.10}$$

and (2.11) yields

$$\partial_w^M \zeta_N(s, w) = (-)^M \prod_{j=0}^{M-1} (s+j) \cdot \zeta_N(s+M, w), \qquad M \in \mathbb{N}^*. \tag{3.11}$$

Turning next to the function

$$\Psi_N(w) \equiv \partial_s \zeta_N(s, w)|_{s=0} \tag{3.12}$$

associated with f (3.1), the representation (2.15) yields

$$\begin{aligned} \Psi_N(w) = {} & \frac{(-)^{N+1}}{N!} B_{N,N}(w) \ln w + (-)^N \sum_{k=0}^{N-1} \frac{B_{N,k}(0)\, w^{N-k}}{k!(N-k)!} \sum_{l=1}^{N-k} \frac{1}{l} \\ & + \sum_{k=N+1}^{M} \frac{(-)^k}{k!} B_{N,k}(0)\, w^{N-k}(k-N-1)! + R_{N,M}(w), \end{aligned} \tag{3.13}$$

with

$$R_{N,M}(w) \equiv \int_0^\infty \frac{dt}{t} e^{-wt} \left(\prod_{j=1}^{N} (1 - e^{-a_j t})^{-1} - \sum_{k=0}^{M} \frac{(-)^k}{k!} B_{N,k}(0)\, t^{k-N} \right), \tag{3.14}$$

where $M \geqslant N$ and $\mathrm{Re}\, w > 0$. From Prop. 2.3 it follows that $\Psi_N(w)$ has a holomorphic extension to $\mathbb{C}^-$ (2.21), and that the remainder in (3.13) satisfies

$$R_{N,M}(w) = O(w^{N-M-1}), \quad |w| \to \infty, \quad |\arg w| < \pi, \tag{3.15}$$

where the bound is uniform for $|\arg w| \leqslant \pi - \delta$, $\delta > 0$. Moreover, (2.17) yields

$$\zeta_N(M, w) = (-)^M \partial_w^M \Psi_N(w)/(M-1)!, \qquad M \geqslant N+1, \tag{3.16}$$

and the integral representation (2.35) becomes

$$\Psi_N(w) = \int_0^\infty \frac{dt}{t}\left(e^{-wt}\prod_{j=1}^N \frac{1}{1-e^{-a_j t}}\right.$$

$$\left. - t^{-N}\sum_{n=0}^{N-1}\frac{(-t)^n}{n!}B_{N,n}(w) - \frac{(-)^N}{N!}e^{-t}B_{N,N}(w)\right). \quad (3.17)$$

To proceed, we introduce the multiple gamma function

$$\Gamma_N(w) \equiv \exp(\Psi_N(w)) = \exp(\partial_s \zeta_N(s, w)|_{s=0}). \quad (3.18)$$

(It should be pointed out that the multiple gamma function $\Gamma_N^B(w)$ defined by Barnes is slightly different: One has

$$\Gamma_N(w) = \Gamma_N^B(w)/\rho_N, \quad (3.19)$$

where ρ_N is Barnes' modular constant. Our definition is in accord with most of the later literature.) Then the recurrence (1.4) entails

$$\Gamma_{M+1}(w \mid a_1, ..., a_{M+1})$$
$$= \Gamma_M(w \mid a_1, ..., a_M)\, \Gamma_{M+1}(w + a_{M+1} \mid a_1, ..., a_{M+1}), \qquad M \in \mathbb{N}, \quad (3.20)$$

with $\Gamma_0(w) \equiv 1/w$.

Next, we recall that $\Psi_{M+1}(w)$ has an analytic continuation to $\mathbb{C}^-$ (2.21). Therefore, $\Gamma_{M+1}(w)$ has an analytic continuation to $\mathbb{C}^-$, too, and has no zeros in $\mathbb{C}^-$. The analytic character of $\Gamma_{M+1}(w)$ for $w \in (-\infty, 0]$ can now be obtained by exploiting (3.20).

Specifically, taking first $M = 0$, one can iterate (3.20) to get

$$\Gamma_1(w \mid a_1) = \prod_{k=0}^{l-1} \frac{1}{w + ka_1} \cdot \Gamma_1(w + la_1 \mid a_1), \qquad l \in \mathbb{N}^*. \quad (3.21)$$

From this one reads off that $\Gamma_1(w \mid a_1)$ has a meromorphic extension without zeros and with simple poles for $w \in -a_1\mathbb{N}$. Writing next

$$\Gamma_2(w \mid a_1, a_2) = \prod_{k=0}^{l-1} \Gamma_1(w + ka_2 \mid a_1) \cdot \Gamma_2(w + la_2 \mid a_1, a_2), \quad l \in \mathbb{N}^*, \quad (3.22)$$

one deduces that $\Gamma_2(w \mid a_1, a_2)$ has a meromorphic extension without zeros and with poles for $w = -(k_1 a_1 + k_2 a_2)$, $k_1, k_2 \in \mathbb{N}$. The multiplicity of a pole w_0 equals the number of distinct pairs (k_1, k_2) such that $w_0 = -(k_1 a_1 + k_2 a_2)$. (In particular, all poles are simple when a_1/a_2 is irrational.)

Proceeding recursively, it is now clear that $\Gamma_N(w)$ has a meromorphic extension, without zeros and with poles for $w = -(k_1 a_1 + \cdots + k_N a_N)$, $k_1, ..., k_N \in \mathbb{N}$. It should be observed that the relations (3.16) between $\zeta_N(M, w)$ (written as the series (3.4)) and the logarithmic derivatives of $\Gamma_N(w)$ are in agreement with these conclusions (though they do not imply them). It should also be noted that the pole of $\Gamma_N(w)$ at $w = 0$ is simple. Denoting its residue by R_N, Barnes' constant ρ_N in (3.19) is (by definition) equal to R_N^{-1}. (Thus one has $w\Gamma_N^B(w) \to 1$ as $w \to 0$.)

To conclude this section, let us consider the $N = 1$ case. From (3.4) we have

$$\zeta_1(s, w \mid a) = a^{-s}\zeta(s, w/a), \tag{3.23}$$

where $\zeta(s, w)$ is the Hurwitz zeta function. Also, (3.13) specializes to

$$\ln \Gamma_1(w \mid a) = \left(\frac{w}{a} - \frac{1}{2}\right) \ln w - \frac{w}{a} + \sum_{k=2}^{M} \frac{(-)^k (w/a)^{1-k} B_k}{k(k-1)}$$
$$+ \int_0^\infty \frac{dx}{x} e^{-wx/a} \left(\frac{1}{1 - e^{-x}} + \sum_{k=0}^{M} \frac{(-x)^{k-1} B_k}{k!}\right), \tag{3.24}$$

where $M \geqslant 1$ and $\operatorname{Re} w > 0$, and where B_k are the Bernoulli numbers, defined by

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{t^k}{k!} B_k. \tag{3.25}$$

Moreover, the integral representation (3.17) can be written as

$$\ln \Gamma_1(w \mid a) = \int_0^\infty \frac{dy}{y} \left(\left(\frac{w}{a} - \frac{1}{2}\right) e^{-2y/a} - \frac{1}{2y} + \frac{e^{-2y(w/a - 1/2)}}{2\operatorname{sh} y} \right). \tag{3.26}$$

Thus we have (see, e.g., Eq. (A37) in Ref. [23], with $z \to w/a - 1/2$)

$$\Gamma_1(w \mid a) = \exp((w/a - 1/2) \ln a)\, \Gamma(w/a)(2\pi)^{-1/2}. \tag{3.27}$$

Finally, we point out that the asymptotics associated with (3.24) amounts to the Stirling series.

4. THE DIFFERENCE EQUATION PERSPECTIVE

We proceed by relating the recurrence relations (1.2) and (1.4) to the general theory of first order analytic difference equations expounded in Appendix A. In this way we obtain simultaneously some illuminating illustrations of this theory and new representations for the pertinent functions. The first question to answer is obviously: In what sense—if any—can ζ_N and Ψ_N be viewed as minimal solutions to difference equations of the form (A.1)?

Comparing (1.2) and (1.4) to (A.1), it is clear that the role of the function ϕ in (A.1) should be played by ζ_M and Ψ_M, resp., and a_{M+1} should be viewed as the step size a. We also need a strip $|\mathrm{Im}\, z| < c$ in which $\phi(z)$ is analytic. Beginning with ζ_{M+1}, let us first define a number

$$A_N \equiv \tfrac{1}{2} \sum_{j=1}^{N} a_j, \qquad N \in \mathbb{N} \tag{4.1}$$

(with $A_0 = 0$). Consider now the function

$$\phi_{M,s}(z) \equiv \zeta_M(s, A_M + d + iz), \qquad d > -A_M, \tag{4.2}$$

where we choose at first $\mathrm{Re}\, s > M$. Because we choose the displacement parameter d greater than $-A_M$, we obtain a non-empty strip $|\mathrm{Im}\, z| < A_M + d$ in which $\phi_{M,s}(z)$ is defined and analytic. Thus we can use (3.2) to write

$$\phi_{M,s}(z) = \frac{2^{1-M}}{\Gamma(s)} \int_0^\infty dy\, \frac{(2y)^{s-1}\, e^{-2dy}}{\prod_{j=1}^{M} \mathrm{sh}(a_j y)} \cdot e^{-2iyz}, \qquad \mathrm{Re}\, s > M,\ \mathrm{Im}\, z < A_M + d. \tag{4.3}$$

Let us now study $\phi_{M,s}(z)$ with regard to the conditions (A.5) of Theorem A.1. The Fourier transform $\hat{\phi}_{M,s}(y)$ (A.4) can be read off from (4.3). It is manifestly in $L^1(\mathbb{R})$ and it satisfies $\hat{\phi}_{M,s}(y) = O(y)$ for $y \to 0$, provided $\mathrm{Re}\, s \geqslant M + 2$. To ensure $\phi_{M,s}(x) \in L^1(\mathbb{R})$ we must require $\mathrm{Re}\, s > M + 1$. (Indeed, this can be readily deduced from the series representation (3.4) for $\phi_{M,s}$.)

Choosing $\mathrm{Re}\, s \geqslant M + 2$, then, Theorem A.1 applies and so we obtain a minimal solution

$$\begin{aligned} f_{M,s}(a_{M+1}; z) &= \frac{2^{-M}}{\Gamma(s)} \int_0^\infty dy\, \frac{(2y)^{s-1}\, e^{-2dy}}{\prod_{j=1}^{M+1} \mathrm{sh}(a_j y)} \cdot e^{-2iyz} \\ &= \zeta_{M+1}(s, A_{M+1} + d + iz),\ \mathrm{Re}\, s \geqslant M+2,\ \mathrm{Im}\, z < A_{M+1} + d \end{aligned} \tag{4.4}$$

to the analytic difference equation

$$f(z+ia_{M+1}/2)-f(z-ia_{M+1}/2)=\phi_{M,s}(z),\ \operatorname{Re}s\geqslant M+2,\ \operatorname{Im}z<A_M+d. \tag{4.5}$$

In words, ζ_{M+1} may be viewed as the unique minimal solution given by Theorem A.1, provided $\operatorname{Re}s\geqslant M+2$.

Next, we consider general s-values. From (3.11) we deduce

$$\partial_z^k\phi_{M,s}(z)=(-i)^k\prod_{j=0}^{k-1}(s+j)\cdot\phi_{M,s+k}(z),\qquad k\in\mathbb{N}^*. \tag{4.6}$$

Thus, fixing s_0 with $s_0\neq 1,\ldots,M+1$, and choosing k_0 such that $\operatorname{Re}s_0+k_0\geqslant M+2$, it follows from the paragraph containing (A.22) that the difference equation (4.5) admits minimal solutions. Recalling the representation (3.8), it readily follows that $\zeta_{M+1}(s_0,A_{M+1}+d+iz)$ is polynomially bounded in the strip $|\operatorname{Im}z|\leqslant a_{M+1}/2$, so that it is once again a *minimal* solution to (4.5). From (A.23) we then obtain the representation (with $N=M+1$)

$$\zeta_N(s_0,A_N+d+iz)=\sum_{j=0}^{k_0}r_{j,s_0}\frac{(iz)^j}{j!}$$
$$+\frac{2^{1-N}}{\Gamma(s_0)}\int_0^\infty dy\,\frac{(2y)^{s_0-1}e^{-2dy}}{\prod_{j=1}^N\operatorname{sh}(a_jy)}\left(e^{-2iyz}-\sum_{j=0}^{k_0-1}\frac{(-2iyz)^j}{j!}\right), \tag{4.7}$$

which holds for $s_0\neq 1,\ldots,N$, $\operatorname{Re}s_0+k_0>N$ and $\operatorname{Im}z<A_N+d$.

Now since $\operatorname{Re}s_0+k_0>N$, we are entitled to evaluate the k_0-fold z-derivative of (4.7) by differentiating k_0 times under the integral sign. From the resulting formula it is readily deduced that the highest coefficient r_{k_0,s_0} vanishes. (Indeed, this follows for instance by comparison with the k_0-fold derivative of (4.4).) Also, the coefficients r_j, $j=0,\ldots,k-1$, in (A.23) cannot readily be expressed in terms of $\eta(z)$, but they are clearly equal to $g^{(j)}(a;0)$. Thus we have

$$r_{k_0,s_0}=0,\qquad r_{j,s_0}=(\partial_w^j\zeta_N)(s_0,A_N+d),\qquad j=0,\ldots,k_0-1 \tag{4.8}$$

in (4.7). It should be noted that the resulting formula can also be directly inferred from (4.4) and analytic continuation in s.

Turning to the difference equation (1.4) obeyed by Ψ_{M+1}, let us consider the function

$$\phi_M(z)\equiv\Psi_M(A_M+d+iz),\qquad d>-A_M. \tag{4.9}$$

Just as $\phi_{M,s}(z)$ (4.2), it is defined and analytic in the non-empty strip $|\mathrm{Im}\, z| < A_M + d$. But it is clear from (3.13) that $\phi_M(z)$ does not satisfy the assumptions of Theorem A.1.

On the other hand, it follows from (3.16) that one has

$$\partial_z^{M+2}\, \phi_M(z) = (-i)^{M+2}\,(M+1)!\, \phi_{M,M+2}(z). \tag{4.10}$$

As we have established above, the rhs satisfies the assumptions of Theorem A.1, so that $\phi_M(z)$ yields an analytic difference equation admitting minimal solutions. Now it is clear from (3.13) that $\Psi_{M+1}(A_{M+1} + d + iz)$ is polynomially bounded for $|\mathrm{Im}\, z| \leqslant a_{M+1}/2$, so it gives rise to a *minimal* solution. Thus we may invoke the general formula (A.23) (using (4.3) with $s = M+2$) to deduce the representation (with $N = M+1$)

$$\begin{aligned}\Psi_N(A_N + d + iz) = {} & \sum_{j=0}^{N} (\partial_w^j \Psi_N)(A_N + d)\,\frac{(iz)^j}{j!} \\ & + 2^{-N} \int_0^\infty \frac{dy}{y}\, \frac{e^{-2dy}}{\prod_{j=1}^N \mathrm{sh}(a_j y)} \left(e^{-2iyz} - \sum_{j=0}^{N} \frac{(-2iyz)^j}{j!} \right),\end{aligned} \tag{4.11}$$

where we may choose $\mathrm{Im}\, z < A_N + d$. (Just as for (4.7), the highest coefficient is readily seen to vanish.)

It should be noted that we used uniqueness of minimal solutions to arrive at this representation. Alternatively, however, it may be derived directly from (4.7) and (4.8) by using that $\Psi_N(w)$ equals (by definition) the s-derivative of $\zeta_N(s, w)$ at $s = 0$.

Quite different-looking representations may be obtained by exploiting the formula (A.7) with $\phi(z)$ given by $\phi_{M,s}(z)$ (4.2) and $\mathrm{Re}\, s \geqslant M+2$. (Note in this connection that this s-restriction entails not only that $\phi_{M,s}(z)$ satisfies the assumptions of Theorem A.1, but also those of Theorem A.2, with c chosen equal to $A_M + d$.) Changing variables, it yields

$$\zeta_{M+1}(s, A_{M+1} + d + iz) = \frac{1}{2ia_{M+1}} \int_{-\infty}^{\infty} dx\, \zeta_M(s, c + iz - ix)\, \mathrm{th}\, \frac{\pi}{a_{M+1}} x, \tag{4.12}$$

where $c = A_M + d$ and where we may take $\mathrm{Im}\, z < c$. Clearly, we can iterate this relation, but before doing so it is expedient to integrate by parts (recall (3.11)):

$$
\begin{aligned}
\zeta_N(s, A_N + d + iz) &= \frac{\pi}{2a_N^2}\frac{1}{s-1}\int_{-\infty}^{\infty} dx\, \zeta_{N-1}(s-1, A_{N-1} + d + iz - ix)\,\mathrm{ch}^{-2}(\pi x/a_N) \\
&= \int_{\mathbb{R}^N}\left(\prod_{n=1}^{N}\frac{\pi\,\mathrm{ch}^{-2}(\pi x_n/a_n)}{2a_n^2(s-n)}\right)\left(d + iz - i\sum_{n=1}^{N} x_n\right)^{N-s} d^N x. \qquad (4.13)
\end{aligned}
$$

(Here, we used $\zeta_0(s, w) = w^{-s}$ in the last iteration step.) As it stands, this new representation is valid for $\mathrm{Re}\, s > N$ and $\mathrm{Im}\, z < d$. But it is plain by inspection that it extends analytically to arbitrary $s \neq 1, \ldots, N$. Moreover, the x_j-contour may be shifted up by $\alpha_j \in (0, a_j/2)$ to enlarge the half plane to $\mathrm{Im}\, z < d + \alpha_j$; more generally, (4.13) can be adjusted so that it holds for a given z_0 with $\mathrm{Im}\, z_0 < A_N + d$.

It is of interest to point out that (4.13) yields an alternative route to an explicit determination of $\zeta_N(-m, A_N + w)$ for $m \in \mathbb{N}$. (Indeed, we also have (3.10) available.) The point is that for $s = -m$ the integrand in (4.13) is a polynomial in $x_1, \ldots, x_n$, so that the integrals can be done by using

$$
\frac{\pi}{2}\int_{-\infty}^{\infty} dy\, \frac{y^{2k}}{\mathrm{ch}^2\, \pi y} = (-)^k\,(2^{1-2k} - 1)\, B_{2k}, \qquad k \in \mathbb{N}, \qquad (4.14)
$$

where B_{2k} are the Bernoulli numbers given by (3.25).

(A short proof of the (known) result (4.14) reads as follows. Denoting the lhs by I_k, the elementary Fourier transform

$$
\frac{\pi}{2}\int_{-\infty}^{\infty} dy\, \frac{\cos xy}{\mathrm{ch}^2\, \pi y} = \frac{x}{2\,\mathrm{sh}(x/2)} \qquad (4.15)
$$

entails

$$
\frac{x e^{x/2}}{e^x - 1} = \sum_{k=0}^{\infty}\frac{(-)^k}{(2k)!} I_k x^{2k}. \qquad (4.16)
$$

But we may also write (cf. (3.25))

$$
\frac{x e^{x/2}}{e^x - 1} = \frac{x}{e^{x/2} - 1} - \frac{x}{e^x - 1} = \sum_{n=0}^{\infty}\frac{x^n}{n!}(2^{1-n} - 1)\, B_n, \qquad (4.17)
$$

so that (4.14) follows upon comparing (4.16) and (4.17).)

From (4.13) we can now quickly obtain the corresponding representation of $\Psi_N(A_N + +d+iz)$, by taking the s-derivative at $s=0$. This yields

$$\Psi_N(A_N+d+iz) = \sum_{l=1}^{N} \frac{1}{l} \cdot \zeta_N(0, A_N+d+iz)$$
$$+(-)^{N+1}\left(\prod_{n=1}^{N} \frac{\pi}{2na_n^2} \int_{-\infty}^{\infty} \frac{dx_n}{\mathrm{ch}^2(\pi x_n/a_n)}\right) I_N(x), \tag{4.18}$$

where the integrand reads

$$I_N(x) = \left(d+iz-i\sum_{n=1}^{N} x_n\right)^N \ln\left(d+iz-i\sum_{n=1}^{N} x_n\right). \tag{4.19}$$

(As before, the restriction $\mathrm{Im}\, z < d$ can be relaxed by suitable contour shifts.)

APPENDIX A

A. *First Order Difference Equations*

This appendix is concerned with analytic difference equations (henceforth AΔEs) of the form

$$f(z+ia/2) - f(z-ia/2) = \phi(z). \tag{A.1}$$

Here, we have $a \in (0, \infty)$ and $\phi(z)$ is a function that is analytic in a strip $|\mathrm{Im}\, z| < c$, $c > 0$, around the real axis. We call a function $f(z)$ a *minimal solution* to the AΔE (A.1) when it has the following properties:

(i) $f(z)$ is analytic in the strip $|\mathrm{Im}\, z| < c + a/2$;

(ii) $f(z)$ satisfies (A.1) in the strip $|\mathrm{Im}\, z| < c$;

(iii) $f(z)$ is polynomially bounded in the strip $|\mathrm{Im}\, z| \leqslant a/2$.

It would be useful to have necessary and sufficient conditions on $\phi(z)$ for minimal solutions to exist, but we are not aware of such conditions. Before turning to conditions that are sufficient for existence, it is important to appreciate why minimal solutions are unique up to a constant, whenever they exist.

To this end, consider the difference

$$d(z) = f_1(z) - f_2(z) \tag{A.2}$$

of two minimal solutions. It is analytic in $|\mathrm{Im}\, z| < c + a/2$ and polynomially bounded in $|\mathrm{Im}\, z| \leqslant a/2$. Since it also satisfies

$$d(z + ia/2) = d(z - ia/2), \qquad |\mathrm{Im}\, z| < c, \tag{A.3}$$

it has an analytic continuation to an entire ia-periodic function. Polynomial boundedness now entails that $d(z)$ is constant.

As concerns necessary conditions, it is clear from (ii) and (iii) that $\phi(x)$, $x \in \mathbb{R}$, must be polynomially bounded as $x \to \pm\infty$. Thus $\phi(x)$ defines a tempered distribution. As such, it admits a Fourier transform in the distributional sense. The following theorem provides sufficient conditions guaranteeing in particular that the Fourier transform

$$\hat{\phi}(y) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, \phi(x)\, e^{ixy} \tag{A.4}$$

exists in the classical sense and yields a continuous function.

THEOREM A.1. *Assuming $\phi(z)$ satisfies*

$$\phi(x) \in L^1(\mathbb{R}), \qquad \hat{\phi}(y) \in L^1(\mathbb{R}), \qquad \hat{\phi}(y) = O(y),\ y \to 0, \tag{A.5}$$

the $A\Delta E$ (A.1) admits minimal solutions. In particular, there exists a minimal solution $f(a; z)$ explicitly given by

$$f(a; z) = \int_{-\infty}^{\infty} dy\, \frac{\hat{\phi}(2y)}{\mathrm{sh}\, ay}\, e^{-2iyz}, \qquad |\mathrm{Im}\, z| \leqslant a/2 \tag{A.6}$$

or by

$$f(a; z) = \frac{1}{2ia} \int_{-\infty}^{\infty} du\, \phi(u)\, \mathrm{th}\, \frac{\pi}{a}(z - u), \qquad |\mathrm{Im}\, z| < a/2. \tag{A.7}$$

This function is bounded for $|\mathrm{Im}\, z| \leqslant a/2$, and satisfies

$$\lim_{x \to \pm\infty} f(a; x + it) = 0, \qquad t \in [-a/2, a/2]. \tag{A.8}$$

Moreover, the following addition formula holds true:

$$f\left(\frac{a}{k}; z\right) = \sum_{j=1}^{k} f\left(a; z + \frac{ia}{2k}(k + 1 - 2j)\right), \qquad |\mathrm{Im}\, z| < c + \frac{a}{2k}. \tag{A.9}$$

Proof. See Theorem II.2 in Ref. [23] and its proof. ∎

We continue by presenting another set of sufficient conditions on $\phi(z)$ that does not involve Fourier transforms. These conditions may be more easily checked in concrete applications.

THEOREM A.2. *Assume $\phi(z)$ is bounded in closed substrips of $|\mathrm{Im}\, z| < c$ and satisfies*

$$\lim_{x \to \pm\infty} \phi(x+it) = 0, \qquad \phi(x+it) \in L^1(\mathbb{R}, dx) \tag{A.10}$$

for all $t \in (-c, c)$. Then the AΔE (A.1) admits minimal solutions. In particular, there exists a minimal solution $f(a; z)$ explicitly given by (A.7). This function is bounded in closed substrips of $|\mathrm{Im}\, z| < c + a/2$ and satisfies

$$\lim_{x \to \pm\infty} f(a; x+it) = \pm \frac{1}{2ia} \int_{-\infty}^{\infty} du\, \phi(u), \quad t \in (-c-a/2, c+a/2). \tag{A.11}$$

Moreover, the addition formula (A.9) holds true.

Proof. Define a function $f(z)$ by the rhs of (A.7). Since $\phi(x) \in L^1(\mathbb{R})$, this function is well defined and analytic for $|\mathrm{Im}\, z| < a/2$. Next, fixing z with $\mathrm{Im}\, z \in (-a/2, a/2)$, we may shift contours to obtain

$$f(z) = \frac{1}{2ia} \int_{-\infty}^{\infty} du\, \phi(u+it)\, \mathrm{th}\, \frac{\pi}{a}(z-u-it), \tag{A.12}$$

provided t satisfies $t \in (-c, c)$ and $\mathrm{Im}\, z - t \in (-a/2, a/2)$. (This readily follows from the assumptions and Cauchy's theorem.)

We can now exploit (A.21) to deduce that $f(z)$ has an analytic continuation to $|\mathrm{Im}\, z| < c + a/2$, once more given by (A.12), where t is such that $\mathrm{Im}\, z - t \in (-a/2, a/2)$ and $t \in (-c, c)$. From this formula one readily sees that $f(z)$ is bounded in closed substrips of $|\mathrm{Im}\, z| < c + a/2$ and obeys (A.11).

We proceed by proving that $f(z)$ satisfies the AΔE (A.1). To this end we fix z with $\mathrm{Im}\, z \in (-c, c)$ and choose $t_\pm$ satisfying

$$t_+ \in \mathrm{Im}\, z + (0, a), \qquad t_- \in \mathrm{Im}\, z + (-a, 0), \qquad t_\pm \in (-c, c). \tag{A.13}$$

Then we may write

$$f\left(z \pm \frac{ia}{2}\right) = \frac{1}{2ia} \int_{-\infty}^{\infty} du\, \phi(u+it_\pm)\, \mathrm{th}\, \frac{\pi}{a}\left(z \pm \frac{ia}{2} - u - it_\pm\right). \tag{A.14}$$

From this we obtain

$$f\left(z+\frac{ia}{2}\right)-f\left(z-\frac{ia}{2}\right)=\frac{1}{2ia}\int_{-\infty}^{\infty} du\,\phi(u+it_+)\,\mathrm{cth}\,\frac{\pi}{a}\left(z-(u+it_+)\right)$$
$$-\frac{1}{2ia}\int_{-\infty}^{\infty} du\,\phi(u+it_-)\,\mathrm{cth}\,\frac{\pi}{a}(z-(u+it_-)). \quad (A.15)$$

Let us now view the rhs as a contour integral

$$\frac{1}{2ia}\int_{\Gamma} dw\,\phi(w)\,\mathrm{cth}\,\frac{\pi}{a}(z-w), \quad (A.16)$$

where Γ is depicted in Fig. 2. Then Cauchy's theorem may be invoked to deduce that the integral equals $-2\pi i$ times the residue at the simple pole $w=z$. Thus the rhs of (A.15) equals $\phi(z)$.

It remains to show that the addition formula (A.9) holds true. Now it is clear that the function on the rhs satisfies the AΔE (A.1) with a replaced by a/k. Since it is also a minimal solution with the same limit for $x\to\infty$ as

$$f\left(\frac{a}{k};x\right)=\frac{k}{2ia}\int_{-\infty}^{\infty} du\,\phi(u)\,\mathrm{th}\,\frac{\pi k}{a}(x-u), \quad (A.17)$$

it must be equal to $f(a/k;z)$, by virtue of uniqueness. ∎

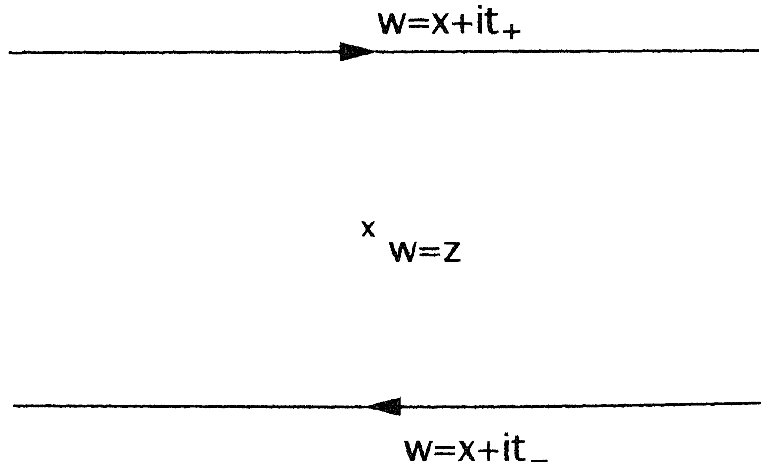

FIG. 2. The contour Γ in the w-plane.

When $\phi(z)$ is such that minimal solutions to (A.1) exist, it is not clear that the derivative $\phi'(z)$ gives rise to an AΔE admitting minimal solutions, too. Of course, when $f(z)$ is a minimal solution to (A.1), it is immediate that $f'(z)$ solves (A.1) with $\phi \to \phi'$, but the point is that the property (iii) may not hold. (Cauchy's integral formula entails $f'(z)$ is polynomially bounded in strips $\text{Im}\, z \in [-a/2+\varepsilon, a/2-\varepsilon]$, $\varepsilon > 0$, but the bound might diverge as $\varepsilon \downarrow 0$.)

By contrast, it is easy to see that primitives of $\phi(z)$ do give rise to AΔEs admitting minimal solutions. Indeed, let

$$\eta'(z) = \phi(z), \qquad |\text{Im}\, z| < c, \tag{A.18}$$

and let $f(z)$ be a minimal solution to (A.1). Setting

$$g(z) \equiv rz + \int_0^z dw\, f(w), \tag{A.19}$$

we now choose r such that the function

$$iar + \int_{z-ia/2}^{z+ia/2} dw\, f(w) \tag{A.20}$$

equals $\eta(z)$. Thus $g(z)$ fulfils

$$g(z+ia/2) - g(z-ia/2) = \eta(z), \qquad |\text{Im}\, z| < c, \tag{A.21}$$

and is obviously a minimal solution to this AΔE.

Of course, this construction can be repeated to handle right-hand side functions $\eta(z)$ satisfying

$$\eta^{(k)}(z) = \phi(z), \qquad k \in \mathbb{N}^*, \qquad |\text{Im}\, z| < c. \tag{A.22}$$

To be specific, when $\phi(z)$ fulfils the assumptions (A.5) of Theorem A.1, one arrives at minimal solutions to (A.21) given by

$$\begin{aligned} g(a; z) = \sum_{j=0}^{k} \frac{r_j z^j}{j!} + \int_{-\infty}^{\infty} dy\, \frac{\hat{\phi}(2y)}{\text{sh}\, ay}(-2iy)^{-k} \\ \cdot \left(e^{-2iyz} - \sum_{j=0}^{k-1} \frac{(-2iyz)^j}{j!} \right), \qquad |\text{Im}\, z| \leqslant a/2, \end{aligned} \tag{A.23}$$

where $r_1, ..., r_k$ are uniquely determined. Indeed, it is clear that the k-fold derivative of the rhs equals $r_k + f(a; z)$, so that $g(a; z)$ satisfies the k-fold derivative of (A.21). The coefficients $r_1, ..., r_k$ are then determined recursively as described in the previous paragraph. (See also Theorem II.3 in

Ref. [23].) Note one has $r_j = g^{(j)}(a; 0)$ for $j = 0, ..., k-1$, but we have no formula expressing r_j and r_k directly in terms of $\eta(z)$.

Assume next (A.22) holds and $\phi(z)$ satisfies the assumptions of Theorem A.2. Then $\eta(z)$ is polynomially bounded in closed substrips of $|\mathrm{Im}\, z| < c$, so the integral

$$I(a; z) \equiv \frac{\pi}{2ia^2} \int_{-\infty}^{\infty} dx \, \frac{\eta(z-x)}{\mathrm{ch}^2(\pi x/a)}, \qquad |\mathrm{Im}\, z| < c \tag{A.24}$$

is well defined and yields a function that is analytic in $|\mathrm{Im}\, z| < c$. A suitable shift of contour then shows that $I(a; z)$ extends analytically to $|\mathrm{Im}\, z| < c + a/2$ (cf. the proof of Theorem A.2). Consider now a function of the form

$$g(a; z) = \sum_{j=0}^{k} \frac{p_j z^j}{j!} + \int_0^z dw \, I(a; w), \qquad |\mathrm{Im}\, z| < c + a/2. \tag{A.25}$$

Clearly, one has

$$g^{(k)}(a; z) = p_k + \frac{\pi}{2ia^2} \int_{-\infty}^{\infty} dx \, \frac{\eta^{(k-1)}(z-x)}{\mathrm{ch}^2(\pi x/a)}, \qquad |\mathrm{Im}\, z| < c. \tag{A.26}$$

Writing

$$\frac{\pi}{a} \frac{1}{\mathrm{ch}^2 \frac{\pi}{a} x} = \partial_x \mathrm{th} \frac{\pi}{a} x, \tag{A.27}$$

we may integrate by parts to deduce that the rhs equals $p_k + c_k + f(a; z)$, cf. (A.7). Therefore, $g^{(k)}(a; z)$ solves the k-fold derivative of the AΔE (A.21). It then follows as before that the coefficients $p_1, ..., p_k$ in (A.25) can be chosen such that $g(a; z)$ solves (A.21), and $g(a; z)$ is clearly minimal.

We close this appendix with a result that is of a less general character, but which is quite relevant for the Barnes multiple zeta functions considered in Section 3. Let us begin by noting that when $\phi(z)$ is analytic in the half plane $\mathrm{Im}\, z < c$, then arbitrary solutions $f(z)$ to (A.1) (in any reasonable sense) satisfy the iterated equation

$$f(z) - f(z - i(N+1)a) = \phi\left(z - \frac{ia}{2}\right) + \phi\left(z - \frac{ia}{2} - ia\right) + \cdots + \phi\left(z - \frac{ia}{2} - iNa\right), \tag{A.28}$$

with $\operatorname{Im} z < c + a/2$. The following theorem yields conditions on $\phi(z)$ (which are stronger than those of Theorem A.2) guaranteeing that the rhs converges as $N \to \infty$, and gives rise to the minimal solution $f(a; z)$ (A.7).

THEOREM 4.3. *Assume that $\phi(z)$ is analytic in $\operatorname{Im} z < c$ and bounded in $\operatorname{Im} z \leqslant c - \varepsilon$ for all $\varepsilon > 0$. Assume that* (A.10) *holds true for all $t \in (-\infty, c)$, and in addition assume*

$$\lim_{t \to -\infty} \int_{-\infty}^{\infty} dx \, |\phi(x + it)| = 0. \tag{A.29}$$

Then the minimal solution $f(a; z)$ (A.7) *is analytic for $\operatorname{Im} z < c + a/2$. Moreover, fixing z with $\operatorname{Im} z < c + a/2$, the series*

$$\sum_{n=0}^{\infty} \phi\left(z - \frac{ia}{2} - ina\right) \tag{A.30}$$

converges and equals $f(a; z)$.

Proof. Just as in the proof of Theorem A.2 it follows that $f(a; z)$ is analytic for $\operatorname{Im} z < c + a/2$ and given by

$$\frac{1}{2ia} \int_{-\infty}^{\infty} du \, \phi(u + it_0) \operatorname{th} \frac{\pi}{a} (z - (u + it_0)), \tag{A.31}$$

where $t_0 \in \operatorname{Im} z + (-a/2, a/2)$ and $t_0 \in (-\infty, c)$. Now we shift the contour $w = u + it_0$, $u \in \mathbb{R}$, to the contour

$$w = u + it_0 - i(N + 1) a, \qquad N \in \mathbb{N}, \tag{A.32}$$

picking up the residues at $w = z - ia/2, ..., z - ia/2 - iNa$. Thus we obtain

$$\begin{aligned} f(a; z) = {} & \sum_{n=0}^{N} \phi\left(z - \frac{ia}{2} - ina\right) \\ & + \frac{1}{2ia} \int_{-\infty}^{\infty} du \, \phi(u + it_0 - i(N + 1) a) \operatorname{th} \frac{\pi}{a} (z - (u + it_0)). \end{aligned} \tag{4.33}$$

Next, we use (A.29) to deduce that the integral has limit 0 for $N \to \infty$. Therefore the series converges, too, and its limit equals $f(a; z)$. ∎

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